

Orthogonal bases of Hermitean monogenic polynomials: an explicit construction in complex dimension 2

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Abstract. In this contribution we construct an orthogonal basis of Hermitean monogenic polynomials for the specific case of two complex variables. The approach combines group representation theory, see [5], with a Fischer decomposition for the kernels of each of the considered Dirac operators, see [4], and a Cauchy-Kovalevskaya extension principle, see [3].

Keywords: Hermitean Clifford analysis, orthogonal basis

PACS: 02.30.-f (MSC 30G35)

BASICS OF HERMITEAN CLIFFORD ANALYSIS

Let (e_1, \dots, e_m) be an orthonormal basis of \mathbb{R}^m , then multiplication in the complex Clifford algebra \mathbb{C}_m is governed by the rule $e_\alpha e_\beta + e_\beta e_\alpha = -2\delta_{\alpha\beta}$, $\alpha, \beta = 1, \dots, m$, whence \mathbb{C}_m is generated additively by the elements $e_A = e_{j_1} \dots e_{j_h}$, where $A = \{j_1, \dots, j_h\} \subset \{1, \dots, m\}$, with $1 \leq j_1 < j_2 < \dots < j_h \leq m$, and $e_\emptyset = 1$.

The framework for Hermitean Clifford analysis is introduced by means of a complex structure, i.e. an $SO(m; \mathbb{R})$ -element J with $J^2 = -1$ (see [1, 2]). So, the dimension is forced to be even: $m = 2n$. Usually J is chosen to act upon the generators of \mathbb{C}_{2n} as $J[e_j] = -e_{n+j}$ and $J[e_{n+j}] = e_j$, $j = 1, \dots, n$. By means of the projection operators $\pm \frac{1}{2}(\mathbf{1} \pm iJ)$ associated to J , first the Witt basis elements $(f_j, f_j^\dagger)_{j=1}^n$ for \mathbb{C}_{2n} are obtained: $f_j = \frac{1}{2}(\mathbf{1} + iJ)[e_j] = \frac{1}{2}(e_j - ie_{n+j})$ and $f_j^\dagger = -\frac{1}{2}(\mathbf{1} - iJ)[e_j] = -\frac{1}{2}(e_j + ie_{n+j})$, $j = 1, \dots, n$, satisfying the relations $f_j f_k + f_k f_j = f_j^\dagger f_k^\dagger + f_k^\dagger f_j^\dagger = 0$ and $f_j f_k^\dagger + f_k^\dagger f_j = \delta_{jk}$, $j, k = 1, \dots, n$. Next, a vector $(x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{R}^{2n}$ is identified with $\underline{X} = \sum_{j=1}^n (e_j x_j + e_{n+j} y_j)$, producing the Hermitean variables $\underline{z} = \frac{1}{2}(\mathbf{1} + iJ)[\underline{X}] = \sum_{j=1}^n f_j z_j$ and $\underline{z}^\dagger = -\frac{1}{2}(\mathbf{1} - iJ)[\underline{X}] = \sum_{j=1}^n f_j^\dagger \bar{z}_j$, expressed in the complex variables $z_j = x_j + iy_j$ and their conjugates $\bar{z}_j = x_j - iy_j$, $j = 1, \dots, n$. Finally, the Dirac operator $\partial_{\underline{X}} = \sum_{j=1}^n (e_j \partial_{x_j} + e_{n+j} \partial_{y_j})$ gives rise to the Hermitean Dirac operators $\partial_{\underline{z}}^\dagger = \frac{1}{4}(\mathbf{1} + iJ)[\partial_{\underline{X}}] = \sum_{j=1}^n f_j \partial_{\bar{z}_j}$ and $\partial_{\underline{z}} = -\frac{1}{4}(\mathbf{1} - iJ)[\partial_{\underline{X}}] = \sum_{j=1}^n f_j^\dagger \partial_{z_j}$, involving the Cauchy–Riemann operators $\partial_{\bar{z}_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ and their conjugates $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$, $j = 1, \dots, n$. The Hermitean variables and Dirac operators are isotropic, whence the Laplacian decomposes as $\Delta_{2n} = 4(\partial_{\underline{z}} \partial_{\underline{z}}^\dagger + \partial_{\underline{z}}^\dagger \partial_{\underline{z}})$, while also $\underline{z} \underline{z}^\dagger + \underline{z}^\dagger \underline{z} = |\underline{z}|^2$.

We take functions with values in an irreducible representation \mathbb{S}_n of \mathbb{C}_{2n} , called spinor space, which is realized within \mathbb{C}_{2n} using a primitive idempotent $I = I_1 \dots I_n$, with $I_j = f_j f_j^\dagger$, $j = 1, \dots, n$. With that choice $\mathbb{S}_n \equiv \mathbb{C}_{2n} I \cong \bigwedge_n^\dagger I$, where \bigwedge_n^\dagger denotes the Grassmann algebra generated by the f_j^\dagger 's, since $f_j I = 0$. Hence \mathbb{S}_n decomposes into homogeneous parts as $\mathbb{S}_n = \bigoplus_{r=0}^n \mathbb{S}_n^{(r)} = \bigoplus_{r=0}^n (\bigwedge_n^\dagger)^{(r)} I$, with $(\bigwedge_n^\dagger)^{(r)} = \text{span}_{\mathbb{C}}(f_{k_1}^\dagger \wedge f_{k_2}^\dagger \wedge \dots \wedge f_{k_r}^\dagger : \{k_1, \dots, k_r\} \subset \{1, \dots, n\})$.

A continuously differentiable function g in an open region Ω of \mathbb{R}^{2n} , taking values in \mathbb{S}_n , then is called (left) Hermitean monogenic in Ω iff it satisfies in Ω the system $\partial_{\underline{z}} g = 0 = \partial_{\underline{z}}^\dagger g$. A major difference with Euclidean Clifford analysis concerns the underlying group invariance. Where $\partial_{\underline{X}}$ is invariant under the action of $SO(m)$, the system invariance of $(\partial_{\underline{z}}, \partial_{\underline{z}}^\dagger)$ breaks down to the group $U(n)$, see e.g. [1, 2]. For this reason $U(n)$ will play a fundamental role in the construction of an orthogonal basis of Hermitean monogenic polynomials, as explained in [5].

The spaces of homogeneous polynomials on \mathbb{C}^n with bidegree of homogeneity (a, b) in $(\underline{z}, \underline{z}^\dagger)$, taking values in $\mathbb{S}_n^{(r)}$, will be denoted by $\mathcal{P}_{a,b}^r(\mathbb{C}^n)$. By $\mathcal{M}_{a,b}(\mathbb{C}^n)$ we denote the space of Hermitean monogenic polynomials of bidegree (a, b) in $(\underline{z}, \underline{z}^\dagger)$, and by $\mathcal{M}_{a,b}^r(\mathbb{C}^n)$ its subspace with values in $\mathbb{S}_n^{(r)}$; the latter may be further split as

$$\mathbb{S}_n^{(r)} \equiv (\bigwedge_n^\dagger)^{(r)} I = (\bigwedge_{n-1}^\dagger)^{(r)} (f_1^\dagger, \dots, f_{n-1}^\dagger) I \bigoplus (\bigwedge_{n-1}^\dagger)^{(r-1)} (f_1^\dagger, \dots, f_{n-1}^\dagger) f_n^\dagger I$$

whence we can decompose polynomials in $\mathcal{M}_{a,b}^r(\mathbb{C}^n)$ as $p_{a,b} = p_{a,b}^0 I + p_{a,b}^1 f_n^\dagger I$, with $p_{a,b}^0$ taking values in $(\Lambda_{n-1}^\dagger)^{(r)}(f_1^\dagger, \dots, f_{n-1}^\dagger)$ and $p_{a,b}^1$ taking values in $(\Lambda_{n-1}^\dagger)^{(r-1)}(f_1^\dagger, \dots, f_{n-1}^\dagger)$. Note that for $r = 0$ or $r = n$ one of these components becomes trivial. In the same order of ideas we single out the variables (z_n, \bar{z}_n) and rewrite the Hermitean variables as $\underline{z} = \tilde{z} + f_n z_n$ and $\underline{z}^\dagger = \tilde{z}^\dagger + f_n^\dagger \bar{z}_n$, and the Hermitean Dirac operators as $\partial_{\underline{z}} = \tilde{\partial}_{\underline{z}} + f_n^\dagger \partial_{z_n}$ and $\partial_{\underline{z}}^\dagger = \tilde{\partial}_{\underline{z}}^\dagger + f_n \partial_{\bar{z}_n}$. We will consider restrictions to $\{z_n = 0 = \bar{z}_n\}$, identified with \mathbb{C}^{n-1} . The following results were then proven in [3].

Proposition 1. (i) Given the polynomial $p_{a,b-j}^0 I \in \text{Ker}(\tilde{\partial}_{\underline{z}})$ on \mathbb{C}^{n-1} ($j = 0, \dots, b$), there exists a unique polynomial $M_{a,b,j}^0 \in \mathcal{M}_{a,b}(\mathbb{C}^n)$, given by

$$M_{a,b,j}^0 = \bar{z}_n^j \left(\sum_{k=0}^{\min(2a+1, 2(b-j))} \frac{1}{\lfloor \frac{k}{2} \rfloor!} \frac{1}{\lfloor \frac{k+1}{2} \rfloor!} \left(z_n \tilde{\partial}_{\underline{z}} f_n + \bar{z}_n \tilde{\partial}_{\underline{z}}^\dagger f_n^\dagger \right)^k p_{a,b-j}^0 I \right)$$

such that $\partial_{\bar{z}_n}^j M_{a,b,j}^0|_{\mathbb{C}^{n-1}} = p_{a,b-j}^0 I$ and all other derivatives w.r.t. \bar{z}_n vanish in \mathbb{C}^{n-1} .

(ii) Given the polynomial $p_{a-i,b}^1 f_n^\dagger I \in \text{Ker}(\tilde{\partial}_{\underline{z}}^\dagger)$ on \mathbb{C}^{n-1} ($i = 0, \dots, a$), there exists a unique polynomial $M_{a,b,i}^1 \in \mathcal{M}_{a,b}(\mathbb{C}^n)$, given by

$$M_{a,b,i}^1 = z_n^i \left(\sum_{k=0}^{\min(2a, 2b+1)} \frac{1}{\lfloor \frac{k}{2} \rfloor!} \frac{1}{\lfloor \frac{k+1}{2} \rfloor!} \left(z_n \tilde{\partial}_{\underline{z}} f_n + \bar{z}_n \tilde{\partial}_{\underline{z}}^\dagger f_n^\dagger \right)^k p_{a-i,b}^1 f_n^\dagger I \right)$$

such that $\partial_{z_n}^i M_{a,b,i}^1|_{\mathbb{C}^{n-1}} = p_{a-i,b}^1 f_n^\dagger I$ and all other derivatives w.r.t. z_n vanish in \mathbb{C}^{n-1} .

The polynomial $M_{a,b,j}^0$ (respectively $M_{a,b,i}^1$) is called the Hermitean Cauchy-Kovalevskaya extension of the initial polynomial $p_{a,b-j}^0 I$ (respectively the initial polynomial $p_{a-i,b}^1 f_n^\dagger I$). This CK extension will play an important role in the construction of the desired orthogonal basis. Indeed, introducing, as in [5], the following spaces of initial polynomials:

$$\begin{aligned} \mathcal{A}_{a,b-j}^r &= \left\{ p_{a-i,b}^0 I \mid p_{a-i,b}^0 I \in \text{Ker}(\tilde{\partial}_{\underline{z}}) \cap \mathcal{P}_{a,b-j}^r(\mathbb{C}^{n-1}) \right\} \\ \mathcal{B}_{a-i,b}^r &= \left\{ p_{a-i,b}^1 f_n^\dagger I \mid p_{a-i,b}^1 I \in \text{Ker}(\tilde{\partial}_{\underline{z}}^\dagger) \cap \mathcal{P}_{a-i,b}^{r-1}(\mathbb{C}^{n-1}) \right\} \end{aligned}$$

the CK extension map is an isomorphism from $\bigoplus_{j=0}^b \mathcal{A}_{a,b-j}^r \oplus \bigoplus_{i=0}^a \mathcal{B}_{a-i,b}^r$ to $\mathcal{M}_{a,b}^r$, commuting with the action of $U(n-1)$, whence it yields a splitting of $\mathcal{M}_{a,b}^r$ into a direct sum of $U(n-1)$ invariant subspaces. Since the initial polynomials on \mathbb{C}^{n-1} for the CK extension have to be submit to the compatibility condition of being either in the kernel of $\tilde{\partial}_{\underline{z}}$ or in the kernel of $\tilde{\partial}_{\underline{z}}^\dagger$, the so-called Fischer decomposition of these kernels in terms of Hermitean monogenics will also be involved. Under the action of $U(n-1)$, see [4], the space $\text{Ker}_{a,b}^r(\tilde{\partial}_{\underline{z}}) \equiv \text{Ker}(\tilde{\partial}_{\underline{z}}) \cap \mathcal{P}_{a,b}^r(\mathbb{C}^{n-1})$ has the multiplicity free irreducible decomposition

$$\text{Ker}_{a,b}^r(\tilde{\partial}_{\underline{z}}) = \mathcal{M}_{a,b}^r \bigoplus_{j=0}^{\min(a,b-1)} |\underline{z}|^{2j} \underline{z}^\dagger \mathcal{M}_{a-j,b-j-1}^{r-1} \bigoplus_{j=0}^{\min(a-1,b-1)} |\underline{z}|^{2j} \left(\underline{z}^\dagger \underline{z} + \frac{(a-j-1+r)}{(a+r)} \underline{z} \underline{z}^\dagger \right) \mathcal{M}_{a-j-1,b-j-1}^r \quad (1)$$

and the space $\text{Ker}_{a,b}^{r-1}(\tilde{\partial}_{\underline{z}}^\dagger) \equiv \text{Ker}(\tilde{\partial}_{\underline{z}}^\dagger) \cap \mathcal{P}_{a,b}^{r-1}(\mathbb{C}^{n-1})$ has the multiplicity free irreducible decomposition

$$\text{Ker}_{a,b}^{r-1}(\tilde{\partial}_{\underline{z}}^\dagger) = \mathcal{M}_{a,b}^{r-1} \bigoplus_{j=0}^{\min(a-1,b)} |\underline{z}|^{2j} \underline{z} \mathcal{M}_{a-j-1,b-j}^r \bigoplus_{j=0}^{\min(a-1,b-1)} |\underline{z}|^{2j} \left(\underline{z} \underline{z}^\dagger + \frac{(b-j-1+n-r+1)}{(b+n-r+1)} \underline{z} \underline{z}^\dagger \right) \mathcal{M}_{a-j-1,b-j-1}^{r-1} \quad (2)$$

It now becomes clear that, once the desired bases have been constructed in dimension $n-1$, these results can be used as building blocks in the above Fischer decompositions, yielding bases for the spaces $\mathcal{A}_{a,b-j}^r$ and $\mathcal{B}_{a-i,b}^r$ of initial polynomials. Subsequent application of the CK extension procedure, will then produce a basis for the space $\mathcal{M}_{a,b}^r$ in dimension n , which, by construction, will be orthogonal w.r.t. any $U(n)$ invariant inner product.

We will now follow this general procedure as explained above, and, in more detail, in [5], to explicitly obtain orthogonal bases for the spaces $\mathcal{M}_{a,b}^r(\mathbb{C}^2)$, $r = 0, 1, 2$, $(a, b) \in \mathbb{N}^2$. Since the procedure is inductive, we need however to start with the case $n = 1$.

THE CASE $n = 1$

In this case we are considering polynomials $f(z_1, \bar{z}_1)$ defined in the complex plane and taking values in the spinor space $\mathbb{S}_1 = \text{span}_{\mathbb{C}}\{1, f_1^\dagger\}I$. The Hermitean Dirac operators are simply $\tilde{\partial}_{\bar{z}} = f_1^\dagger \partial_{z_1}$ and $\tilde{\partial}_{\bar{z}}^\dagger = f_1 \partial_{\bar{z}_1}$, whence Hermitean monogenicity means nothing else but anti-holomorphy in the case $r = 0$ and holomorphy in the case $r = 1$. The symmetry group here is $U(1) \simeq SO(2)$.

For $r = 0$ the $U(1)$ modules $\tilde{\mathcal{M}}_{0,b}^0$ are given by $\text{span}_{\mathbb{C}}\left\{\frac{\bar{z}_1^b}{b!}I\right\}$, $b = 0, 1, 2, \dots$. They have highest weight $(-b)$.

For $r = 1$ the $U(1)$ modules $\tilde{\mathcal{M}}_{a,0}^1$ are given by $\text{span}_{\mathbb{C}}\left\{\frac{z_1^a}{a!}f_1^\dagger I\right\}$, $a = 0, 1, 2, \dots$. They have highest weight $(a+1)$.

THE CASE $n = 2$

Now we consider polynomials $f(z_1, \bar{z}_1, z_2, \bar{z}_2)$ taking values in the spinor space $\mathbb{S}_2 = \text{span}_{\mathbb{C}}\{1, f_1^\dagger, f_2^\dagger, f_1^\dagger f_2^\dagger\}I$. If $r = 0$ or $r = 2$ we are again confronted with (anti-)holomorphy, see [2], so we will focus on the interesting case $r = 1$.

The dimension of the $U(2)$ module $\mathcal{M}_{a,b}^1$ is $a+b+2$, see [3]. Each of the spaces of initial polynomials $\mathcal{A}_{a,b-j}^1$, $j = 0, \dots, b$ and $\mathcal{B}_{a-i,b}^1$, $i = 0, \dots, a$, is one-dimensional. The general theory of the CK extension procedure, see [3], predicts that the compatibility conditions imposed on these initial polynomials will be trivially fulfilled, so they simply are all homogeneous polynomials in the variables z_1 and \bar{z}_1 of the appropriate bidegree, which is moreover confirmed by the Fischer decompositions (1)–(2):

$$\begin{aligned}\mathcal{A}_{a,b-j}^1 &= \text{span}_{\mathbb{C}}\left\{(-1)^{b-j} \frac{z_1^a}{a!} \frac{\bar{z}_1^{b-j}}{(b-j)!} f_1^\dagger I\right\}, & j = 0, \dots, b \\ \mathcal{B}_{a-i,b}^1 &= \text{span}_{\mathbb{C}}\left\{(-1)^b \frac{z_1^{a-i}}{(a-i)!} \frac{\bar{z}_1^b}{b!} f_2^\dagger I\right\}, & i = 0, \dots, a\end{aligned}$$

By CK extension each of the spaces of initial polynomials thus gives rise to exactly one Hermitean monogenic basis polynomial, together yielding an orthogonal basis for $\mathcal{M}_{a,b}^1$, see [5]. These basis polynomials are respectively given by

$$\begin{aligned}M_{a,b,j}^0 &= \sum_{k=0}^{\min(a,b-j)} (-1)^{b-j-k} \frac{z_2^k}{k!} \frac{\bar{z}_2^{k+j}}{(k+j)!} \frac{z_1^{a-k}}{(a-k)!} \frac{\bar{z}_1^{b-j-k}}{(b-j-k)!} f_1^\dagger I \\ &\quad + \sum_{k=0}^{\min(a,b-j-1)} (-1)^{b-j-k-1} \frac{z_2^k}{k!} \frac{\bar{z}_2^{k+j+1}}{(k+j+1)!} \frac{z_1^{a-k}}{(a-k)!} \frac{\bar{z}_1^{b-j-k-1}}{(b-j-k-1)!} f_2^\dagger I, & j = 0, \dots, b \\ M_{a,b,i}^1 &= \sum_{k=0}^{\min(a-i,b)} (-1)^{b-k} \frac{z_2^{k+i}}{(k+i)!} \frac{\bar{z}_2^k}{k!} \frac{z_1^{a-i-k}}{(a-i-k)!} \frac{\bar{z}_1^{b-k}}{(b-k)!} f_2^\dagger I \\ &\quad + \sum_{k=0}^{\min(a-i-1,b)} (-1)^{b-k} \frac{z_2^{k+i+1}}{(k+i+1)!} \frac{\bar{z}_2^k}{k!} \frac{z_1^{a-i-k-1}}{(a-i-k-1)!} \frac{\bar{z}_1^{b-k}}{(b-k)!} f_1^\dagger I, & i = 0, \dots, a\end{aligned}$$

The following properties may then be verified right away.

Property 1. Under derivation with respect to the "new" variables (z_2, \bar{z}_2) , the orthogonal basis polynomials of $\mathcal{M}_{a,b}^1$ act as follows:

$$\begin{aligned}\partial_{z_2} M_{a,b,i}^1 &= M_{a-1,b,i-1}^1 & \partial_{\bar{z}_2} M_{a,b,i}^1 &= M_{a,b-1,i+1}^1 & i = 1, \dots, a \\ \partial_{z_2} M_{a,b,j}^0 &= M_{a-1,b,j+1}^0 & \partial_{\bar{z}_2} M_{a,b,j}^0 &= M_{a,b-1,j-1}^0 & j = 1, \dots, b \\ \partial_{z_2} M_{a,b,0}^1 &= M_{a-1,b,0}^0 & \partial_{\bar{z}_2} M_{a,b,0}^0 &= M_{a,b-1,0}^1\end{aligned}$$

Property 2. Under derivation with respect to the "old" variables (z_1, \bar{z}_1) , the orthogonal basis polynomials of $\mathcal{M}_{a,b}^1$ act as follows:

$$\begin{aligned}\partial_{z_1} M_{a,b,i}^1 &= M_{a-1,b,i}^1 & -\partial_{\bar{z}_1} M_{a,b,i}^1 &= M_{a,b-1,i}^1 & i = 0, \dots, a \\ \partial_{z_1} M_{a,b,j}^0 &= M_{a-1,b,j}^0 & -\partial_{\bar{z}_1} M_{a,b,j}^0 &= M_{a,b-1,j}^0 & j = 0, \dots, b\end{aligned}$$

Remark 1. Property 1 holds in any dimension n , whereas Property 2 is specific for the case $n = 2$.

Remark 2. Since the orthogonal Hermitean monogenic basis polynomials are determined only up to a constant, the final expressions may always be normalized, according to some preferred behaviour or property. Here, we have in fact normalized all initial data by requiring that

- (i) if $p_{a,b}^0 I \in \mathcal{A}_{a,b}^1$, then $\partial_{z_1}^a (-\partial_{\bar{z}_1})^b [p_{a,b}^0 I] = f_1^\dagger I$;
- (ii) if $p_{a,b}^1 f_2^\dagger I \in \mathcal{B}_{a,b}^1$, then $\partial_{z_1}^a (-\partial_{\bar{z}_1})^b [p_{a,b}^1 f_2^\dagger I] = f_2^\dagger I$.

These normalization conditions are reflected in the eventual orthogonal basis as follows:

$$\partial_{z_1}^a (-\partial_{\bar{z}_1})^{b-j} [M_{a,b,j}^0] = \frac{\bar{z}_2^j}{j!} f_1^\dagger I, \quad \partial_{z_1}^{a-i} (-\partial_{\bar{z}_1})^b [M_{a,b,i}^1] = \frac{z_2^i}{i!} f_2^\dagger I$$

Finally let us check the branching rules for the space $\mathcal{M}_{a,b}^1$ with highest weight $\lambda = (a+1, -b)$. From group representation theory, see e.g. [6], we know that when restricting the symmetry to $U(1)$, the irreducible $U(2)$ module $\mathcal{M}_{a,b}^1$ decomposes into irreducible $U(1)$ modules as

$$\mathcal{M}_{a,b}^1 = \bigoplus_{\mu \succ \lambda} V_\mu = \bigoplus_{k=-b}^{a+1} V_k$$

where each summand appears with multiplicity one; this decomposition is orthogonal w.r.t. any $U(2)$ invariant scalar product on $\mathcal{M}_{a,b}^1$. On the other hand the Fischer decompositions (1)–(2) produce the $U(1)$ irreducible components of the spaces of initial data $\mathcal{A}_{a,b-j}^1$ ($j = 0, \dots, b$) and $\mathcal{B}_{a-i,b}^1$ ($i = 0, \dots, a$). Assuming that $a > b$ (the cases $a \leq b$ being similar) we have in fact that $\mathcal{A}_{a,b-j}^1$ is a shifted version of the $U(1)$ module $\widetilde{\mathcal{M}}_{a-b+j,0}^1$ with highest weight $(a-b+j+1)$, for all $j = 0, \dots, b$. Similarly, $\mathcal{B}_{a-i,b}^1$ is a shifted version of the $U(1)$ module $\widetilde{\mathcal{M}}_{a-i-b-1,0}^1$ with highest weight $(a-i-b)$, for all $i = 0, \dots, a-b-1$, while $\mathcal{B}_{a-i,b}^1$ is a shifted version of $\widetilde{\mathcal{M}}_{0,b-a+i}^0$ with highest weight $(a-i-b)$, for all $i = a-b, \dots, a$. As the CK extension map is an isomorphism between the initial data space $\bigoplus_{j=0}^b \mathcal{A}_{a,b-j}^1 \oplus \bigoplus_{i=0}^a \mathcal{B}_{a-i,b}^1$ and the space $\mathcal{M}_{a,b}^1$, which commutes with the action of $U(1)$, our construction of the orthogonal basis of $\mathcal{M}_{a,b}^1$ exactly yields the above splitting of $\mathcal{M}_{a,b}^1$ into the direct sum of $a+b+2$ $U(1)$ invariant subspaces V_k , $k = -b, \dots, a+1$.

Remark 3. For completeness we mention here the cases $r = 0$ and $r = 2$. For $r = 0$ the orthogonal basis of $\mathcal{M}_{0,b}^0$ consists of all homogeneous anti-holomorphic polynomials in (\bar{z}_1, \bar{z}_2) , i.e. $\frac{\bar{z}_1^{b-j} \bar{z}_2^j}{(b-j)! j!}$, $j = 0, \dots, b$, while for $r = 2$, the orthogonal basis of $\mathcal{M}_{a,0}^2$ consists of all homogeneous holomorphic polynomials in (z_1, z_2) , i.e. $\frac{z_1^{a-i} z_2^i}{(a-i)! i!}$, $i = 0, \dots, a$.

ACKNOWLEDGMENTS

R. Lávička and V. Souček acknowledge support by grant MSM 0021620839 and by grant GA CR 201/08/0397.

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